B.Sc. II, SEM IV, Differential Equations (1)

UNIT-IV

Homogenous Linear Equations with Constant Coefficients:

These are equations of the type:

$$\frac{\partial^n z}{\partial x^n} + A_1 \frac{\partial^n z}{\partial x^{n-1}\partial y} + A_2 \frac{\partial^n z}{\partial x^{n-2}\partial y^2} + \cdots + A_n \frac{\partial^n z}{\partial y^n} = f(x,y) \longrightarrow (1)$$

where $A_1, A_2, \cdots, A_n$ are constants.

Let $D \equiv \frac{\partial}{\partial x}$ and $D' \equiv \frac{\partial}{\partial y}$. Then

$$(D^n + A_1 D^{n-1} D' + A_2 D^{n-2} D'^2 + \cdots + A_n D'^n) z = f(x,y)$$

Let $\phi(D, D') \equiv \sum_{r=0}^{n} A_r D^{n-r} D'^r$ where $A_0 = 1$.

Then $\phi(D,D') z = f(x,y) \longrightarrow (2)$

The solution of such an equation consists of two parts: the complementary function (C.F.) and the particular integral (P.I.).

Finding the C.F.

The C.F. is a solution of the equation

$$\phi(D, D') z = 0 \longrightarrow (3)$$

Let $u = y + mx$ and let $z = g(u)$ be ~~the~~ a solution of (3).

$Dz = \frac{\partial z}{\partial x} = \frac{dz}{du}\frac{du}{dx} = m g'(u)$

$D^2 z = m^2 g''(u)$

$\vdots$

$D^n z = m^n g^{(n)}(u)$

$D'z = \frac{dz}{du}\frac{du}{dy} = g'(u)$

$D'^2 z = g''(u)$

$\vdots$

$D'^n z = g^{(n)}(u)$

$$\Rightarrow D^{n-r} D'^r z = D^{n-r} g^{(n)}(u) = m^{n-r} g^{(n)}(u)$$

$$(3) \Rightarrow \phi(D,D') z = \sum_{r=0}^{n} A_r D^{n-r} D'^r z$$

$$= \sum_{r=0}^{n} A_r m^{n-r} g^{(n)}(u) = 0$$

$$\Rightarrow m^n + A_1 m^{n-1} + A_2 m^{n-2} + \cdots + A_n = 0 \longrightarrow (4)$$

This is called the Auxillary Equation (A.E.)

It is obtained by replacing D by m and D' by 1 in $\phi(D, D') = 0$.

If $m_1, m_2, \dots, m_n$ are distinct roots of A.E.,
then $C.F. = g_1(y+m_1x) + g_2(y+m_2x) + \dots + g_n(y+m_nx)$.
If the A.E. has two equal roots $m_1 = m_2 = m$, then $(D-mD')^2$ is a factor of $\phi(D,D')$.

$\Rightarrow (D-mD')^2 z = 0$

Let $(D-mD')z = G$

$\Rightarrow (D-mD')G = 0$

$\Rightarrow \frac{\partial G}{\partial x} - m\frac{\partial G}{\partial y} = 0$

so the subsidiary equations are

$$\frac{dx}{1} = \frac{dy}{-m} = \frac{dG}{0}$$

$\Rightarrow y + mx = c$ and $G = g_1(c) = g_1(y+mx)$.

$\Rightarrow (D-mD')z = G = g_1(y+mx)$

$$\Rightarrow \frac{dx}{1} = \frac{dy}{-m} = \frac{dz}{g_1(y+mx)}$$

$\Rightarrow y+mx = c'$ and $dz = g_1(c')x$

$\Rightarrow z = x g_1(c') + g_2(c')$

$= x g_1(y+mx) + g_2(y+mx)$.

Similarly if m is a root repeated k times,
then $C.F. = x^{k-1} g_1(y+mx) + x^{k-2} g_2(y+mx) + \dots + g_k(y+mx)$.

<u>Ex</u> $(D^3 - D^2D' - DD'^2 + D'^3)z = 0$

A.E. $= m^3 - m^2 - m + 1 = 0$

$\Rightarrow (m+1)(m^2 - 2m + 1) = 0$

$\Rightarrow m = -1, 1, 1$

$\Rightarrow z = g_1(y-x) + x g_2(y+x) + g_3(y+x)$.

<u>Finding the Particular Integral (P.I.)</u>

$$P.I. = \frac{1}{\phi(D,D')} f(x,y)$$

This can be obtained by factoring $\phi(D,D')$, by resolving it into partial fractions or by expanding it into an infinite series:

Ex $(D^2+6DD'+9D'^2)z = 12x^2+36xy$

A.E. $m^2+6m+9=0$

$\Rightarrow m=-3,-3$

C.F. $= x\,g_1(y-3x) + g_2(y-3x)$

$$P.I. = \frac{1}{(D+3D')^2}\,12(x^2+3xy)$$

$$= \frac{12}{D^2}\left(1+3\frac{D'}{D}\right)^{-2}(x^2+3xy)$$

$$= \frac{12}{D^2}\left(1-6\frac{D'}{D}+3.3\frac{D'^2}{D^2}-\cdots\right)(x^2+3xy)$$

$$= \frac{12}{D^2}\left(x^2+3xy-6\frac{1}{D}.3x\right)$$

$$= \frac{12}{D^2}(x^2-3xy-9x^2) = \frac{12}{D^2}(-8x^2-3xy)$$

$$= \frac{12}{D}\left(-\frac{8}{3}x^3-\frac{3}{2}x^2y\right) = 12\left(-\frac{8}{3}\frac{x^4}{4}-\frac{3}{2}\frac{x^3}{3}y\right)$$

$$= -12\left(\frac{2}{3}x^4+\frac{x^3y}{2}\right)$$

$\therefore z = x\,g_1(y-3x)+g_2(y-3x)-8x^4-6x^3y$

Some methods of finding P.I.

①. Let $\phi(D,D')z = \cancel{f(ax+by)}\; g(x,y)$

$D f(ax+by) = a f'(ax+by)$	$D' f(ax+by) = b f'(ax+by)$
$D^2 f(ax+by) = a^2 f''(ax+by)$	$D'^2 f(ax+by) = b^2 f''(ax+by)$
$\vdots$	$\vdots$
$D^n f(ax+by) = a^n f^{(n)}(ax+by)$	$D'^n f(ax+by) = b^n f^{(n)}(ax+by)$

$$\Rightarrow \phi(D,D')f(ax+by) = \sum_{r=0}^{n} A_r D^{n-r} D'^r f(ax+by)$$

$$= \sum_{r=0}^{n} A_r a^{n-r} b^r f^{(n)}(ax+by)$$

$$= \phi(a,b) f^{(n)}(ax+by)$$

$$\Rightarrow \frac{1}{\phi(D,D')} f^{(n)}(ax+by) = \frac{1}{\phi(a,b)} f(ax+by)$$

provided that $\phi(a,b) \neq 0$.

If $\phi(a,b) = 0$, then $(bD-aD')$ is a factor of $\phi(D,D')$.

Let $(bD-aD')z = x^r f(ax+by)$

$$\Rightarrow b\frac{\partial z}{\partial x} - a\frac{\partial z}{\partial y} = x^r f(ax+by)$$

$$\Rightarrow \frac{dx}{b} = \frac{dy}{-a} = \frac{dz}{x^r f(ax+by)}$$

$$\Rightarrow ax+by = c \quad \& \quad b\,dz = x^r f(c)\,dx$$

$$\Rightarrow bz = \frac{x^{r+1}}{r+1} f(ax+by)$$

So if $(bD-aD')^k z = f(ax+by)$, then

$$z = \frac{1}{(bD-aD')^{k-1}\,(bD-aD')} f(ax+by)$$

$$= \frac{1}{(bD-aD')^{k-2}\,(bD-aD')} \frac{x}{b} f(ax+by)$$

$$= \frac{1}{(bD-aD')^{k-3}\,(bD-aD')} \frac{x^2}{2b^2} f(ax+by)$$

$$\vdots$$

$$= \frac{x^k}{k!\,b^k} f(ax+by)$$

(2) $\phi(D,D')z = x^m y^l$

$$z = \frac{1}{\phi(D,D')} x^m y^l$$

Take D^n or D'^n common from $\phi(D,D')$ and expand $\phi(D,D')^{-1}$ into an infinite series

if $m \leq l$, then take D'^n common in $\phi(D,D')$

if $m \geq l$, then take D^n common in $\phi(D,D')$.

(3) General method: Write $\phi(D,D') = (D-m_1D')(D-m_2D')\cdots(D-m_nD')$

$$\frac{1}{\phi(D,D')} f(x,y) = \frac{1}{(D-m_1D')(D-m_2D')\cdots(D-m_nD')} f(x,y)$$

We first obtain $u = \frac{1}{D-mD'} f(x,y)$

$\Rightarrow \frac{\partial u}{\partial x} - m\frac{\partial u}{\partial y} = f(x,y)$

$\Rightarrow \frac{dx}{1} = \frac{dy}{-m} = \frac{du}{f(x,y)}$

$\Rightarrow y + mx = c$ and $u = \int f(x,y)\,dx$
$= \int f(x, c-mx)\,dx$

Some general P.I.'s are:

$\frac{1}{\phi(D,D')} e^{ax+by} = \frac{1}{\phi(a,b)} e^{ax+by}$, if $\phi(a,b) \neq 0$

$\frac{1}{\phi(D^2, D'^2, DD')} \sin(ax+by) = \frac{1}{\phi(-a^2,-b^2,-ab)} \sin(ax+by)$

If $\phi(-a^2, -b^2, -ab) \neq 0$

$\frac{1}{\phi(D^2, D'^2, DD')} \cos(ax+by) = \frac{1}{\phi(-a^2,-b^2,-ab)} \cos(ax+by)$

If $\phi(-a^2, -b^2, -ab) \neq 0$.

$\frac{1}{\phi(D,D')} x^m y^l = [\phi(D,D')]^{-1} x^m y^l$

$\frac{1}{\phi(D,D')} (e^{ax+by} V) = e^{ax+by} \frac{1}{\phi(D+a, D'+b)} V.$

Ex $(D^2 - 5DD' + 4D'^2)z = \sin(4x+y)$.

A.E. $m^2 - 5m + 4 = 0 \Rightarrow m = 1, 4$.

C.F. $g_1(y+x) + g_2(y+4x)$

P.I. $= \frac{1}{D^2 - 5DD' + 4D'^2} \sin(4x+y)$

If we put $D^2 = -4^2$, $DD' = -4.1$, $D'^2 = -1^2$, then
$D^2 - 5DD' + 4D'^2 = 0$

$= \frac{1}{(D-4D')(D-D')} \sin(4x+y)$

$\frac{1}{D-D'} \sin(4x+y) = \frac{D+D'}{D^2-D'^2} \sin(4x+y)$

$= \frac{D+D'}{-16+1} \sin(4x+y) = -\frac{1}{15}(4\cos(4x+y) + \cos(4x+y))$

$= \frac{-1}{3} \cos(4x+y)$

P.I. $= \frac{-1}{3} \frac{1}{D-4D'} \cos(4x+y)$

Let $\frac{1}{D-4D'} \cos(4x+y) = u$

$\Rightarrow \frac{\partial u}{\partial x} - 4\frac{\partial u}{\partial y} = \cos(4x+y)$

$\Rightarrow \frac{dx}{1} = \frac{dy}{-4} = \frac{du}{\cos(4x+y)}$

$\Rightarrow y+4x=c$ and $du = \cos c \cdot dx$

$= x \sin c = x \cos(4x+y)$

$\Rightarrow$ P.I $= -\frac{1}{3} x \cos(4x+y)$

$z = g_1(y+x) + g_2(y+4x) - \frac{1}{3} x \cos(4x+y).$

Equations reducible to the homogenous form:

If the equation has terms of the form

$x^n \frac{\partial^n z}{\partial x^n}, \quad x^{n-1} \frac{\partial^{n-1} z}{\partial x^{n-1}}, \quad \ldots, \quad x^2 \frac{\partial^2 z}{\partial x^2}, \quad x\frac{\partial z}{\partial x}$ etc.

then it can be reduced to the homogenous form by putting $x = e^u$ and $y = e^t$

$\Rightarrow \frac{dx}{du} = e^u = x$ & $\frac{dy}{dt} = e^t = y$

Let $D_1 \equiv \frac{\partial}{\partial u}$ and $D_2 \equiv \frac{\partial}{\partial t}$

$\Rightarrow Dz = \frac{\partial z}{\partial x} = \frac{\partial z}{\partial u}\frac{du}{dx} = \frac{1}{x}\frac{\partial z}{\partial u} = \frac{1}{x} D_1 z$

& $D'z = \frac{\partial z}{\partial y} = \frac{\partial z}{\partial t}\frac{dt}{dy} = \frac{1}{y}\frac{\partial z}{\partial t} = \frac{1}{y} D_2 z$

$\Rightarrow D \equiv \frac{1}{x} D_1$ and $D' \equiv \frac{1}{y} D_2$

$\Rightarrow D^2 z = -\frac{1}{x^2} D_1 z + \frac{1}{x}\left(\frac{1}{x} D_1(D_1 z)\right) = \frac{1}{x^2} D_1(D_1 - 1)z$

$D^3 z = \frac{-2}{x^3} D_1(D_1-1)z + \frac{1}{x^2}\frac{1}{x} D_1(D_1^2 - D_1)z$

$= \frac{1}{x^3} D_1 (D_1-1)(D_1-2)z \; \ldots$

Similarly $D'^2 z = \frac{1}{y^2} D_2(D_2-1)z$

$D'^3 z = \frac{1}{y^3} D_2 (D_2-1)(D_2-2)z \; \ldots$

$DD'z = D\frac{1}{y} D_2 z = \frac{1}{y}\frac{1}{x} D_1 D_2 z, \ldots$

Non-homogenous equations with constant coefficients:

These are equations of the type

$$(D-m_1D'-\alpha_1)(D-m_2D'-\alpha_2)\cdots(D-m_kD'-\alpha_k)z=0$$

where $\alpha_1, \alpha_2, \ldots, \alpha_k$ are constants

Let $(D-mD'-\alpha)z=0$

$$\Rightarrow \frac{\partial z}{\partial x} - m\frac{\partial z}{\partial y} = \alpha z$$

$$\Rightarrow \frac{dx}{1} = \frac{dy}{-m} = \frac{dz}{\alpha z}$$

$$\Rightarrow y+mx=c \quad \text{and} \quad \frac{dz}{z} = \alpha dx$$

$$\Rightarrow z = c' e^{\alpha x} = g(y+mx)\, e^{\alpha x}$$

$$\Rightarrow z = e^{\alpha_1 x} g_1(y+m_1x) + e^{\alpha_2 x} g_2(y+m_2x) + \cdots + e^{\alpha_k x} g_k(y+m_kx).$$

For repeated factors :

$$(D-mD'-\alpha)^l z = 0$$

$$\Rightarrow z = e^{\alpha x} g_1(y+mx) + x e^{\alpha x} g_2(y+mx) + \cdots + x^{l-1} e^{\alpha x} g_l(y+mx)$$

Notations: $p=\frac{\partial z}{\partial x}$, $q=\frac{\partial z}{\partial y}$, $r=\frac{\partial^2 z}{\partial x^2}$, $s=\frac{\partial^2 z}{\partial x\partial y}$, $t=\frac{\partial^2 z}{\partial y^2}$

Ex $x^2 r - y^2 t + px - qy = \ln x$

$$\Rightarrow (x^2D^2 - y^2D'^2 + xD - yD')z = \ln x$$

$x = e^u$, $y = e^t$

$D_1 \equiv \frac{\partial}{\partial u}$ $\quad D_2 \equiv \frac{\partial}{\partial t}$

$$\Rightarrow (D_1(D_1-1) - D_2(D_2-1) + D_1 - D_2)z = u$$

$$\Rightarrow (D_1^2 - D_2^2)z = u$$

A.E. $\quad m^2-1=0 \quad m=\pm 1$

C.F. $= g_1(u+t) + g_2(\cancel{t}-u)$

$= g_1(\ln y + \ln x) + g_2(\ln y - \ln x)$

$= h_1(xy) + h_2(y/x)$

$$\text{P.I.} = \frac{1}{D_1^2 - D_2^2}u = \frac{1}{D_1^2}\left(1-\frac{D_2^2}{D_1^2}\right)^{-1}u$$

$$= \frac{1}{D_1^2}\left(1+\frac{D_2^2}{D_1^2}+\cdots\right)u$$

$$= \frac{1}{D_1^2}u = \frac{1}{D_1}\frac{u^2}{2} = \frac{1}{6}u^3 = \frac{(\ln x)^3}{6}$$

$$\boxed{z = h_1(xy) + h_2(y/x) + \frac{(\ln x)^3}{6}}$$

Ex $\quad s + p - q = z + xy$

$\Rightarrow (DD' + D - D' - 1) z = xy$

$\Rightarrow (D-1)(D'+1) z = xy$

C.F. $= e^{x} g_1(y) + e^{-y} g_2(x)$

This is because if $(D'+1) z = 0$

$\Rightarrow \frac{dx}{0} = \frac{dy}{1} = \frac{dz}{-z}$

$\Rightarrow x = c$ and $dz = -z\,dy$

$\Rightarrow z = g_2(x)\, e^{-y}$

P.I. $= \frac{1}{(D-1)(D'+1)} xy$

$= -(1-D)^{-1}(1+D')^{-1} xy$

$= -(1-D)^{-1}(1 - D' + D'^2 + \cdots) xy$

$= -(1-D)^{-1}(xy - x)$

$= -(1 + D + D^2 + \cdots)(xy - x)$

$= -(xy - x + y - 1)$

$\therefore \quad z = e^{x} g_1(y) + e^{-y} g_2(x) - xy + x - y + 1.$

$\longleftrightarrow$.